Tutorial 7

Exercise 1. *Use the simplex method to solve the two-person zero-sum game with game matrix*

$$\begin{pmatrix} 3 & 1 & -2 \\ 2 & 3 & -1 \\ -1 & -2 & 2 \end{pmatrix}.$$

Solution. Step 1. Add 2 to each entry, we get

$$\begin{pmatrix} 5 & 3 & 0 \\ 4 & 5 & 1 \\ 1 & 0 & 4 \end{pmatrix}.$$

Step 2. Set up the tableau as

	y_1	y_2	y_3	
x_1	5	3	0	1
x_2	4	5	1	1
x_3	1	0	4	1
	1	1	1	0

.

Step 3. Apply pivoting operations, we have

	y_1	y_2	y_3	
x_1	5	3	0	1
x_2	4	5	1	1
x_3	1	0	4^*	1
	1	1	1	0

$\rightarrow$

	y_1	y_2	x_3	
x_1	5	3	0	1
x_2	$\frac{15}{4}$	5^*	$-\frac{1}{4}$	$\frac{3}{4}$
y_3	$\frac{1}{4}$	0	$\frac{1}{4}$	$\frac{1}{4}$
	$\frac{3}{4}$	1	$-\frac{1}{4}$	$-\frac{1}{4}$

$\rightarrow$

$$
\begin{array}{rc|ccc|c}
 & & y_1 & x_2 & x_3 & \\
\hline
 & x_1 & & -\frac{3}{5} & & \frac{11}{20} \\
\rightarrow & y_2 & \frac{3}{4} & \frac{1}{5} & -\frac{1}{20} & \frac{3}{20} \\
 & y_3 & & 0 & & \frac{1}{4} \\
\hline
 & & 0 & -\frac{1}{5} & -\frac{1}{5} & -\frac{2}{5}
\end{array}.
$$

Let $d = \frac{2}{5}$. Then the value of the game is $v = \frac{1}{d} - 2 = \frac{1}{2}$. Since the basic solution is

$$
\begin{aligned}
x_3 &= \frac{1}{5} \\
x_1 &= 0 \\
y_3 &= \frac{1}{4} \\
y_2 &= \frac{3}{20} \\
x_2 &= \frac{1}{5} \\
y_1 &= 0.
\end{aligned}
$$

We have the maximin strategy for the row player is

$$\boldsymbol{p} = \frac{1}{d}(x_1, x_2, x_3) = \frac{5}{2}(0, \frac{1}{5}, \frac{1}{5}) = (0, \frac{1}{2}, \frac{1}{2}),$$

and the minimax strategy for the column player is

$$\boldsymbol{q} = \frac{1}{d}(y_1, y_2, y_3) = \frac{5}{2}(0, \frac{3}{20}, \frac{1}{4}) = (0, \frac{3}{8}, \frac{5}{8}).$$

Exercise 2. *Let A be an $m \times n$ matrix. Let*

$$C = \mathrm{conv}(\{\mathrm{a_1}, \cdots, \mathrm{a_n}, \mathrm{e_1}, \cdots, \mathrm{e_m}\})$$

be the convex hull of set $\{\boldsymbol{a}_1, \cdots, \boldsymbol{a}_n, \boldsymbol{e}_1, \cdots, \boldsymbol{e}_m\}$*, where* $\boldsymbol{a}_1^T, \cdots, \boldsymbol{a}_n^T$ *are the column vectors of* A *and* $\boldsymbol{e}_1, \cdots, \boldsymbol{e}_m$ *are the vectors in the standard basis of* $\mathbb{R}^m$*. Prove if* C *contains a point* $(c, \cdots, c) \in \mathbb{R}^m$ *with* $c \leq 0$*, then the value of* A*,* $v(A) \leq c$*.*

Proof. Since $(c, \cdots, c) \in C$, there exist $\lambda_1, \cdots, \lambda_{n+m}$ such that

$$\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n + \lambda_{n+1} \boldsymbol{e}_1 + \cdots + \lambda_{n+m} \boldsymbol{e}_m = (c, \cdots, c),$$

where $0 \leq \lambda_i \leq 1$ and $\lambda_1 + \cdots + \lambda_{n+m} = 1$.

Since $c \leq 0$, at least one of $\lambda_1, \cdots, \lambda_n$ is positive. Multiply both sides of the above equation by $\frac{1}{\lambda_1 + \cdots + \lambda_n}$, we have

$$\frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n} \boldsymbol{a}_1 + \cdots + \frac{\lambda_n}{\lambda_1 + \cdots + \lambda_n} \boldsymbol{a}_n = \left(\frac{c - \lambda_{n+1}}{\lambda_1 + \cdots + \lambda_n}, \cdots, \frac{c - \lambda_{n+m}}{\lambda_1 + \cdots + \lambda_n}\right).$$

Taking transpose, we have

$$(\boldsymbol{a}_1^T \quad \cdots \quad \boldsymbol{a}_n^T) \begin{pmatrix} \frac{\lambda_1}{\lambda_1 + \cdots + \lambda_n} \\ \vdots \\ \frac{\lambda_n}{\lambda_1 + \cdots + \lambda_n} \end{pmatrix} = \frac{1}{\lambda_1 + \cdots + \lambda_n} \begin{pmatrix} c - \lambda_{n+1} \\ \cdots \\ c - \lambda_{n+m} \end{pmatrix}.$$

Note that $A = (\boldsymbol{a}_1^T, \cdots, \boldsymbol{a}_n^T)$. Write $\boldsymbol{q} = \frac{1}{\lambda_1 + \cdots + \lambda_n}(\lambda_1, \cdots, \lambda_n)$. Then $\boldsymbol{q} \in \mathcal{P}^n$ and

$$\boldsymbol{x} A \boldsymbol{q}^T \leq \max_{1 \leq i \leq m} \frac{c - \lambda_{n+i}}{\lambda_1 + \cdots + \lambda_n} \leq c, \text{ since } c \leq 0.$$

Hence

$$v(A) = \min_{\boldsymbol{y} \in \mathcal{P}^n} \max_{\boldsymbol{x} \in \mathcal{P}^m} \boldsymbol{x} A \boldsymbol{y}^T \leq \max_{x \in \mathcal{P}^m} \boldsymbol{x} A \boldsymbol{q}^T \leq c.$$